

Singular behaviour of the electromagnetic field (the static case revisited)

C. Vrejoiu[‡] and R. Zus[§]

University of Bucharest, Department of Physics,
PO Box MG - 11, Bucharest-Magurele, RO - 077125, Romania

Abstract. The singularities of the electromagnetic field are derived to include all the point-like multipoles representing an electric charge and current distribution. Partial results obtained in a previous paper [1] are completed to represent accurately all the terms included in these singularities.

1. Introduction

In the cases of electrostatic and magnetostatic fields of point-like dipoles, one has the well-known procedure of introducing Dirac δ -function terms for obtaining correct expressions of the electric and magnetic fields defined on the entire space. The corresponding field expressions take the following form [2]:

$$\mathbf{E}_p(\mathbf{r}) = -\frac{1}{3\varepsilon_0} \mathbf{p} \delta(\mathbf{r}) + \frac{1}{4\pi\varepsilon_0} \frac{3(\boldsymbol{\nu} \cdot \mathbf{p})\boldsymbol{\nu} - \mathbf{p}}{r^3} = -\frac{1}{3\varepsilon_0} \mathbf{p} \delta(\mathbf{r}) + (\mathbf{E})_{r \neq 0}, \quad (1)$$

where $\boldsymbol{\nu} = \mathbf{r}/r$, and

$$\mathbf{B}_m(\mathbf{r}) = \frac{2\mu_0}{3} \mathbf{m} \delta(\mathbf{r}) + \frac{\mu_0}{4\pi} \frac{3(\boldsymbol{\nu} \cdot \mathbf{m})\boldsymbol{\nu} - \mathbf{m}}{r^3} = \frac{2\mu_0}{3} \mathbf{m} \delta(\mathbf{r}) + (\mathbf{B})_{r \neq 0}. \quad (2)$$

In these equations, by $(\dots)_{r \neq 0}$ we understand an expression in which the derivatives are calculated supposing $r \neq 0$, representing some well-known expressions of the fields. The expressions from equations (1) and (2) are introduced in Ref. [2] as conditions of compatibility with the average value of the electric or magnetic field over a spherical domain containing all the charges or currents inside. Another procedure for introducing equations (1) and (2) is based on an extension of the derivative $\partial_i \partial_j / (1/r)$ to the entire space [3]:

$$\partial_i \partial_j \frac{1}{r} = -\frac{4\pi}{3} \delta_{ij} \delta(\mathbf{r}) + \frac{3\nu_i \nu_j - \delta_{ij}}{r^3}. \quad (3)$$

A more pedagogical and suitable approach for understanding the origin of the difference between the electric and magnetic cases is done in Ref. [4]. Refs. [5] and [6] contain

[‡] E-mail : vrejoiu@fizica.unibuc.ro

[§] E-mail: roxana.zus@fizica.unibuc.ro

generalizations of the equations (1) and (2) to the dynamic case for oscillating electric and magnetic dipoles.

The objective of the present paper is to establish the singularities of the electromagnetic field associated to a system of electric charges and currents assimilated with a point-like multipolar system. These singularities are established for an arbitrary multipolar order in the static case.

In section 2, the procedure of separating the δ -form singularities of the multiple partial derivatives of r^{-1} is described. In sections 3 and 4, we separate the δ -form singularities of the fields corresponding to the point-like equivalent multipole distributions, based on the multipole expansions of the fields associated to an electric charge or current distribution confined in a domain \mathcal{D} .

We point out that the formalism presented in this paper has as mathematical basis the properties of the irreducible tensorial representations of the proper rotations group [7]. For pedagogical and larger accessibility reasons, we give an explicit calculation based on the properties of the tensor contractions such that the procedure has a simple algebraic character. A counterpart of the procedure used in the present paper could be represented by the technique of the spherical function expansions and the cited issues can be a basis for such an approach. However, the use of the spherical coordinates can lead to the omission of the δ -form singularity contributions, as argued in the conclusions.

2. Some delta-function identities

The treatment of some delta-function identities in Ref. [3] can be easily generalized to obtain the necessary identities for higher order derivatives of $1/r$. In the points different from the origin O , the function $1/r$ is a solution of the Laplace equation:

$$\Delta \frac{1}{r} = 0, \quad r \neq 0. \quad (4)$$

It is well-known how one can extend $\Delta(1/r)$ as a distribution to the entire space, as solution of the Poisson equation:

$$\Delta \frac{1}{r} = -4\pi \delta(\mathbf{r}). \quad (5)$$

Equation (3) represents the next step of generalization and can be continued for arbitrary multiple partial derivatives of $1/r$ i.e. for $\partial_{i_1} \dots \partial_{i_n} (1/r)$ for arbitrary n . For $r \neq 0$, this multiple partial derivative is given by the formula

$$\left(\partial_{i_1} \dots \partial_{i_n} \frac{1}{r} \right)_{r \neq 0} = \frac{1}{r^{n+1}} C_{i_1 \dots i_n}^{(n)}. \quad (6)$$

In this equation, the coefficient C is fully symmetric with respect to $i_1 \dots i_n$ and can be written as

$$C_{i_1 \dots i_n}^{(n)} = \sum_k^{\left[\frac{n}{2}\right]} (-1)^k (2n - 2k - 1)!! \delta_{\{i_1 i_2 \dots i_{2k-1} i_{2k}\}} \nu_{2k+1} \dots \nu_{i_n}. \quad (7)$$

In the last equation, $[\beta]$ is the integer part of β and by $A_{\{i_1 \dots i_n\}}$ we understand the sum over all the permutations of the symbols i_q giving distinct terms. From equation (5), it

is obvious that we have to consider the derivatives of $1/r$ extended over the entire space as distributions (generalized functions) which contain singular distributions (δ -functions for example) as separate terms.

Let be a function $F(\mathbf{r})$ and suppose the existence of the integral of the product $F(\mathbf{r}) \phi(\mathbf{r})$, with $\phi(\mathbf{r})$ an arbitrary smooth function (a test function from the domain of the distributions), on the spherical region \mathcal{D}_R , with arbitrary radius R , delimited by the spherical surface Σ_R with the center in O :

$$\int_{\mathcal{D}_R} d^3x F(\mathbf{r}) \phi(\mathbf{r}) = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < r < R} d^3x F(\mathbf{r}) \phi(\mathbf{r}) . \quad (8)$$

This integral can be expressed excluding from the domain \mathcal{D}_R a spherical domain of radius ε centered in O . Writing this last limit of integrals, we can interpret the function $F(\mathbf{r})$ as a distribution defined by

$$\langle (F(\mathbf{r}))_{r \neq 0}, \phi(\mathbf{r}) \rangle = \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_R \setminus \mathcal{D}_\varepsilon} d^3x (F(\mathbf{r}))_{r \neq 0} \phi(\mathbf{r}) .$$

This distribution can be extended such that its support includes the point O . A new term $\theta(\varepsilon - r) F(\mathbf{r})$ can be naturally introduced by the identity

$$F_\varepsilon(\mathbf{r}) = \theta(\varepsilon - r) F(\mathbf{r}, t) + \theta(r - \varepsilon) F(\mathbf{r}),$$

associated with the extension of the integral to the entire domain \mathcal{D}_R :

$$\langle F(\mathbf{r}), \phi(\mathbf{r}) \rangle = \lim_{\varepsilon \rightarrow 0} \left[\int_{\mathcal{D}_\varepsilon} d^3x F(\mathbf{r}) \phi(\mathbf{r}) + \int_{\mathcal{D}_R \setminus \mathcal{D}_\varepsilon} d^3x F(\mathbf{r}) \phi(\mathbf{r}) \right] . \quad (9)$$

Moreover, we suppose the existence of the integral (8) for the partial derivatives of F . Let us consider the partial derivative $\partial_i F(\mathbf{r}, t)$ and the problem of extending this function as a distribution. The definition (9) becomes:

$$\begin{aligned} \langle F(\mathbf{r}, t), \phi(\mathbf{r}) \rangle = \lim_{\varepsilon \rightarrow 0} & \left[\oint_{\Sigma_\varepsilon} dS \nu_i F(\mathbf{r}, t) \phi(\mathbf{r}) - \int_{\mathcal{D}_\varepsilon} d^3x F(\mathbf{r}, t) \partial_i \phi(\mathbf{r}) \right. \\ & \left. + \int_{\mathcal{D}_R \setminus \mathcal{D}_\varepsilon} d^3x \partial_i F(\mathbf{r}, t) \phi(\mathbf{r}) \right] , \end{aligned} \quad (10)$$

where Σ_ε is the sphere of radius ε centered in O and the Gauss theorem was employed. Let us apply this definition to the derivative $\partial_i \partial_j (1/r)$ and, for simplifying the notation, let

$$D_{i_1 \dots i_n}(\mathbf{r}, t) = \partial_{i_1} \dots \partial_{i_n} \frac{1}{r},$$

such that we can write

$$(D_{ij}, \phi) = \lim_{\varepsilon \rightarrow 0} \left[\oint_{\Sigma_\varepsilon} dS \nu_i \partial_j \frac{1}{r} \phi(\mathbf{r}) - \int_{\mathcal{D}_\varepsilon} d^3x \partial_j \frac{1}{r} \partial_i \phi(\mathbf{r}) + \int_{\mathcal{D}_R \setminus \mathcal{D}_\varepsilon} d^3x (\partial_i \partial_j \frac{1}{r}) \phi(\mathbf{r}) \right] \quad (11)$$

Since the last integral on the domain $\mathcal{D}_R \setminus \mathcal{D}_\varepsilon$ represents the distributions associated with the F - expressions for $r \neq 0$ and, in the case of the electromagnetic field, they will be the well-known expressions of the multipole expansions, in the following, we consider only that part of $\langle D_{ij} \rangle$ containing singular distributions with point-like support i.e.,

actually, the difference

$$\begin{aligned} \langle (D_{ij})_{(0)}, \phi \rangle &= \langle D_{ij}, \phi \rangle - \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_R \setminus \mathcal{D}_\varepsilon} d^3x \left(\partial_i \partial_j \frac{1}{r} \right) \phi(\mathbf{r}) \\ &= \lim_{\varepsilon \rightarrow 0} \left[\oint_{\Sigma_\varepsilon} dS \nu_i \partial_j \frac{1}{r} \phi(\mathbf{r}) - \int_{\mathcal{D}_\varepsilon} d^3x \partial_j \frac{1}{r} \partial_i \phi(\mathbf{r}) \right]. \end{aligned} \quad (12)$$

By $D(\mathbf{r}_0)$ we denote a distribution having as support the point given by the vector \mathbf{r}_0 . The surface integral,

$$\lim_{\varepsilon \rightarrow 0} \oint_{\Sigma_\varepsilon} dS \nu_i \partial_j \frac{1}{r} \phi(\mathbf{r}) = - \lim_{\varepsilon \rightarrow 0} \oint_{\Sigma_\varepsilon} dS \frac{1}{r^2} \nu_i \nu_j \phi(\mathbf{r}),$$

after inserting the Taylor series of the function $\phi(\mathbf{r})$ and since on the sphere $r = \varepsilon$, becomes [3],

$$\lim_{\varepsilon \rightarrow 0} \oint_{\Sigma_\varepsilon} dS \nu_i \partial_j \frac{1}{r} \phi(\mathbf{r}) = - \lim_{\varepsilon \rightarrow 0} \int d\Omega(\boldsymbol{\nu}) \nu_i \nu_j [\phi(0) + \varepsilon \nu_k (\partial_k \phi)_0 + \dots] \quad (13)$$

Let us introduce the angular average:

$$\langle g(\boldsymbol{\nu}) \rangle = \frac{1}{4\pi} \int g(\boldsymbol{\nu}) d\Omega(\boldsymbol{\nu}). \quad (14)$$

Particularly, we have the well-known formula [8]:

$$\langle \nu_{i_1} \dots \nu_{i_n} \rangle = \begin{cases} 0, & n = 2k + 1, \\ \frac{1}{(n+1)!!} \delta_{\{i_1 i_2 \dots i_{n-1} i_n\}}, & n = 2k, \quad k = 0, 1, \dots \end{cases} \quad (15)$$

Excepting the term containing $\phi(0)$, all the terms in equation (14) are proportional to positive powers of ε and, consequently, vanish with $\varepsilon \rightarrow 0$, such that

$$\lim_{\varepsilon \rightarrow 0} \oint_{\Sigma_\varepsilon} dS \nu_i \partial_j \frac{1}{r} \phi(\mathbf{r}) = -4\pi \langle \nu_i \nu_j \rangle \phi(0) = -\frac{4\pi}{3} \delta_{ij} \phi(0) = -\frac{4\pi}{3} \delta_{ij} \langle \delta(\mathbf{r}), \phi(\mathbf{r}) \rangle.$$

Considering the second integral in the right-hand side of equation (11), we can write

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d^3x \partial_j \frac{1}{r} \partial_i \phi(\mathbf{r}) = - \lim_{\varepsilon \rightarrow 0} \int_0^\varepsilon r^2 dr \int d\Omega(\boldsymbol{\nu}) \nu_j \frac{1}{r^2} [(\partial_i \phi)_0 + r \nu_k (\partial_i \partial_k \phi)_0 + \dots] = 0,$$

such that finally

$$\left(\partial_i \partial_j \frac{1}{r} \right)_{(0)} = -\frac{4\pi}{3} \delta_{ij} \delta(\mathbf{r}), \quad (16)$$

i.e. the delta-singularity from equation (3) of Ref. [3].

Let us consider the distribution D_{ijk} . Considering only the part having O as support,

$$\langle (D_{ijk})_{(0)}, \phi \rangle = \lim_{\varepsilon \rightarrow 0} \left[\oint_{\Sigma_\varepsilon} dS \nu_i \partial_j \partial_k \frac{1}{r} \phi(\mathbf{r}) - \int_{\mathcal{D}_\varepsilon} d^3x \left(\partial_j \partial_k \frac{1}{r} \right) \partial_i \phi(\mathbf{r}) \right]. \quad (17)$$

The surface integral becomes:

$$\lim_{\varepsilon \rightarrow 0} \oint_{\Sigma_\varepsilon} dS \nu_i \partial_j \partial_k \frac{1}{r} \phi(\mathbf{r}) = \lim_{\varepsilon \rightarrow 0} \oint_{\Sigma_\varepsilon} dS \nu_i \frac{1}{r^3} (3\nu_j \nu_k - \delta_{jk}) \phi(\mathbf{r})$$

and, introducing the Taylor series for $\phi(\mathbf{r})$,

$$\lim_{\varepsilon \rightarrow 0} \oint_{\Sigma_\varepsilon} dS \nu_i \partial_j \partial_k \frac{1}{r} \phi(\mathbf{r}) = 4\pi \lim_{\varepsilon \rightarrow 0} \left\langle \frac{1}{\varepsilon} (3\nu_i \nu_j \nu_k - \nu_i \delta_{jk}) [\phi(0) + \varepsilon \nu_l (\partial_l \phi)_0 + \dots] \right\rangle, \quad (18)$$

and

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \oint_{\Sigma_\varepsilon} dS \nu_i \partial_j \partial_k \frac{1}{r} \phi(\mathbf{r}) \\ &= 4\pi \langle 3\nu_i \nu_j \nu_k \nu_l - \nu_i \nu_l \delta_{jk} \rangle (\partial_l \phi)_0 = 4\pi \left(\frac{1}{5} \delta_{\{ij} \delta_{kl\}} - \frac{1}{3} \delta_{il} \delta_{jk} \right) (\partial_l \phi)_0 . \end{aligned} \quad (19)$$

Concerning the integral on \mathcal{D}_ε from equation (17), we have to observe that, beginning from this derivative order, there is a non-zero contribution for $\varepsilon \rightarrow 0$ [3]. Indeed, introducing equation (16) in equation (17) and noticing that the term $(\partial_j \partial_k (f(\tau)/r))_{r \neq 0}$ gives a null contribution to the limit for $\varepsilon \rightarrow 0$, we can write

$$-\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d^3x \partial_j \partial_k \frac{1}{r} \partial_i \phi(\mathbf{r}) = \frac{4\pi}{3} \int_{\mathcal{D}_\varepsilon} d^3x \delta_{jk} \delta(\mathbf{r}) \partial_i \phi(\mathbf{r}) = \frac{4\pi}{3} \delta_{jk} (\partial_i \phi)_0 .$$

Finally, equations (17), (18) and (19) give

$$(D_{ijk})_{(0)} = -\frac{4\pi}{5} \delta_{\{ij} \partial_k \delta(\mathbf{r}) , \quad (20)$$

i.e. the delta-singularity from equation (4) of Ref. [3].

Obviously, this procedure becomes very complicated for higher order derivatives. Fortunately, for the electromagnetic field, some invariance properties allow a considerable simplification of such calculations.

3. Singularities of the electrostatic field

Let us consider the multipole expansions of the electrostatic field. Given an electric charge distribution with support included in the domain \mathcal{D} , the scalar potential is expressed in the exterior of a sphere containing this domain by the following multipolar series:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{n \geq 0} \frac{(-1)^n}{n!} \partial_{i_1} \dots \partial_{i_n} \frac{\mathbf{P}_{i_1 \dots i_n}}{r} = \frac{1}{4\pi\varepsilon_0} \sum_{n \geq 0} \frac{(-1)^n}{n!} \nabla^n || \frac{\mathbf{P}^{(n)}}{r} . \quad (21)$$

In this expansion, the coordinate system origin O is supposed in \mathcal{D} and $\mathbf{P}^{(n)}$ is the n -th order electric multipolar moment defined by the Cartesian components in the general dynamic case:

$$\mathbf{P}_{i_1 \dots i_n}(t) = \int_{\mathcal{D}} d^3x x_{i_1} \dots x_{i_n} \rho(\mathbf{r}, t) : \quad \mathbf{P}^{(n)}(t) = \int_{\mathcal{D}} d^3x \mathbf{r}^n \rho(\mathbf{r}, t). \quad (22)$$

In equation (21) we employed the following notation for tensorial contractions:

$$(\mathbf{A}^{(n)} || \mathbf{B}^{(m)})_{i_1 \dots i_{|n-m|}} = \begin{cases} A_{i_1 \dots i_{n-m} j_1 \dots j_m} B_{j_1 \dots j_m} & , n > m \\ A_{j_1 \dots j_n} B_{j_1 \dots j_n} & , n = m \\ A_{j_1 \dots j_n} B_{j_1 \dots j_n i_1 \dots i_{m-n}} & , n < m \end{cases} . \quad (23)$$

For the multipole expansion of the electric field $\mathbf{E}(\mathbf{r}) = -\nabla \Phi(\mathbf{r})$, we can write

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla^{n+1} || \frac{\mathbf{P}^{(n)}}{r} = \frac{1}{4\pi\varepsilon_0} \mathbf{e}_i \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \partial_i \partial_{i_1} \dots \partial_{i_n} \frac{\mathbf{P}_{i_1 \dots i_n}}{r}, \quad (24)$$

where, for simplicity, the electric charged system is considered neutral ($Q = 0$). We have to search the singularities of \mathbf{E} given by equations (24). It appears that cumbersome

calculations are involved for higher n if we apply the formulae for higher order derivatives of $1/r$ as in the previous section. However, we can employ an invariance property of the electrostatic field to the substitutions of all moments $\mathbf{P}^{(n)}$, for all n , by their corresponding symmetric and trace-free **STF** projections $\mathcal{P}^{(n)}$ [9, 10, 11]. Retaining the notation \mathbf{p} for the first order moment, this invariance stands for the invariance of the multipole expansion of the electrostatic field to the following substitutions:

$$\mathbf{p}, \mathbf{P}^{(2)}, \mathbf{P}^{(3)}, \dots \rightarrow \mathbf{p}, \mathcal{P}^{(2)}, \mathcal{P}^{(3)}, \dots ; \quad (25)$$

These **STF** tensors can be expressed by the following formula:

$$\mathcal{P}_{i_1 \dots i_n} = \frac{(-1)^n}{(2n-1)!!} \int_{\mathcal{D}} d^3x \rho(\mathbf{r}) \mathbf{r}^{2n+1} \partial_{i_1} \dots \partial_{i_n} \frac{1}{r}, \quad (26)$$

which, actually, differ from the projections by numerical factors. Here,

$$\partial_{i_1 \dots i_n}^{(\lambda)} = \partial_{i_1} \dots \partial_{i_{\lambda-1}} \partial_{i_{\lambda+1}} \dots \partial_{i_n} .$$

Some care is necessary when one considers this invariance property when we have to establish the delta-type singularities of the electromagnetic field, since this property is true only for $r \neq 0$. Indeed, as we firstly see in the electrostatic case, the substitutions $\mathbf{P}^{(n)} \rightarrow \mathcal{P}^{(n)}$ in equation (24) give additional terms containing $\Delta(1/r)$ and their derivatives, which, in the case $r \neq 0$, can be eliminated. But, extending the expressions associated to the multipole expansions to the entire space, including the origin O , these additional terms give notable contributions. This process of searching the delta-form singularities of the electromagnetic field is lost in the previous paper [1] and here, we try to correct it.

Let us consider the delta-singularity corresponding to the electric dipolar field:

$$\mathbf{E}_{(0)}^{(1)} = \frac{1}{4\pi\epsilon_0} \mathbf{e}_i \partial_i \partial_j \frac{p_j}{r} = -\frac{1}{3\epsilon_0} \mathbf{p} \delta(\mathbf{r}), \quad (27)$$

a result obtained by directly applying equation (16).

For the 4-polar term from \mathbf{E} , we firstly consider the expansion (24) expressed by the primitive moments $\mathbf{P}^{(n)}$:

$$\mathbf{E}^{(2)}(\mathbf{r}) = -\frac{1}{8\pi\epsilon_0} \mathbf{e}_i \partial_i \partial_j \partial_k \frac{P_{jk}}{r}. \quad (28)$$

With the help of equation (20), we obtain:

$$\begin{aligned} \mathbf{E}_{(0)}^{(2)} &= \frac{1}{10\epsilon_0} \mathbf{e}_i P_{jk} \delta_{\{ij} \partial_{k\}} \delta(\mathbf{r}) = \frac{1}{10\epsilon_0} \mathbf{e}_i P_{jk} (\delta_{ij} \partial_k \delta(\mathbf{r}) + \delta_{ik} \partial_j \delta(\mathbf{r}) + \delta_{jk} \partial_i \delta(\mathbf{r})) \\ &= \frac{1}{10\epsilon_0} \mathbf{e}_i (2P_{ij} \partial_j \delta(\mathbf{r}) + P_{jj} \partial_i \delta(\mathbf{r})) = \frac{1}{5\epsilon_0} \mathbf{e}_i \left(P_{ij} \partial_j \delta(\mathbf{r}) + \frac{1}{2} P_{jj} \partial_i \delta(\mathbf{r}) \right). \end{aligned} \quad (29)$$

Note that $\mathbf{P}^{(2)}$ is symmetric. This result suggests the possibility to facilitate the calculation of the delta-type singularities of the electromagnetic field if instead of the “primitive” tensors $\mathbf{P}^{(n)}$ we can employ the **STF** projections $\mathcal{P}^{(n)}$. For the present case, it will be simpler to calculate the contractions of the last type of tensors with the coefficients $C^{(n)}$ from equation (6).

For $n = 2$, we search the trace-free part of the symmetric tensor $\mathbf{P}^{(2)}$ as

$$\mathbf{P}_{ij} = \mathcal{P}_{ij} + \delta_{ij} \Lambda . \quad (30)$$

The parameter Λ is determined such that $\mathcal{P}_{ii} = 0$. One obtains

$$\Lambda = \frac{1}{3} \mathbf{P}_{ii} . \quad (31)$$

Let us introduce equation (30) in equation (28):

$$\mathbf{E}^{(2)}(\mathbf{r}) = -\frac{1}{8\pi\epsilon_0} \mathbf{e}_i \mathcal{P}_{jk} \partial_i \partial_j \partial_k \frac{1}{r} - \frac{\Lambda}{8\pi\epsilon_0} \mathbf{e}_i \partial_i \left(\Delta \frac{1}{r} \right) .$$

For $r \neq 0$, the last term vanishes and, indeed, \mathbf{E} is invariant to the substitution $\mathcal{P}^{(2)} \rightarrow \mathbf{P}^{(2)}$. Extending this field to the entire space, this last term contributes with a delta-singularity such that we must write

$$\mathbf{E}^{(2)}(\mathbf{r}) = -\frac{1}{8\pi\epsilon_0} \mathbf{e}_i \mathcal{P}_{jk} \partial_i \partial_j \partial_k \frac{1}{r} + \frac{1}{2\epsilon_0} \Lambda \nabla \delta(\mathbf{r}) . \quad (32)$$

Inserting equation (20) in equation (32), and retaining only the delta-singularities, we can write

$$\mathbf{E}_{(0)}^{(2)}(\mathbf{r}) = \frac{1}{5\epsilon_0} \mathbf{e}_i \mathcal{P}_{ij} \partial_j \delta(\mathbf{r}) + \frac{1}{2\epsilon_0} \Lambda \nabla \delta(\mathbf{r}) . \quad (33)$$

Substituting equation (30) in equation (33), one obtains equation (29).

The advantage of employing the **STF** moments $\mathcal{P}^{(n)}$ instead of the primitive moments $\mathbf{P}^{(n)}$ is manifest for higher order terms from the multipolar expansion. Even from $n = 3$, the contraction of the angular average of a product of more than six factors ν with a primitive moment represented by a tensor which is only symmetric becomes cumbersome.

Though, maybe, only of theoretical interest, let us consider the general case of arbitrary n . The **STF** projection $\mathcal{T}(\mathcal{P}^{(n)})$ of the symmetric tensor $\mathbf{P}^{(n)}$ is defined, up to a numerical factor, by the equation

$$\mathcal{T}_{i_1 \dots i_n}(\mathbf{P}^{(n)}) \equiv \mathcal{P}_{i_1 \dots i_n} = \mathbf{P}_{i_1 \dots i_n} - \delta_{\{i_1 i_2} \Lambda_{i_3 \dots i_n\}} , \quad (34)$$

where $\Lambda^{(n-2)}$ is a symmetric tensor and is defined by the conditions of the trace-free character of $\mathcal{P}^{(n)}$. For low values of n (the ones of practical interest), the components $\Lambda_{i_1 \dots i_n}$ can be calculated directly from the equation system representing the vanishing of all the traces of the tensor $\mathcal{P}^{(n)}$. For higher orders n , there is a general formula known in literature [8], [12] which, with the notation from the present paper, is written as

$$[\mathcal{T}[\mathbf{P}^{(n)}]]_{i_1 \dots i_n} = \sum_{m=0}^{[n/2]} \frac{(-1)^m (2n-1-2m)!!}{(2n-1)!!} \delta_{\{i_1 i_2 \dots i_{2m-1} i_{2m}} \mathbf{P}_{i_{2m+1} \dots i_n}^{(n:m)} . \quad (35)$$

$\mathbf{P}_{i_{2m+1} \dots i_n}^{(n:m)}$ denotes the components of the $(n-2m)$ -th order tensor obtained from $\mathbf{P}^{(n)}$ by contracting m pairs of symbols i . This equation is known as the *detracer theorem* [12]. As a consequence of this theorem, the components of the tensor $\Lambda^{(n-2)}$ are written as

$$\Lambda_{i_1 \dots i_{n-2}}[\mathbf{P}^{(n)}] = \sum_{m=0}^{[n/2-1]} \frac{(-1)^m [2n-1-2(m+1)]!!}{(m+1)(2n-1)!!} \delta_{\{i_1 i_2 \dots i_{2m-1} i_{2m}} \mathbf{P}_{i_{2m+1} \dots i_{n-2}}^{(n:m+1)} . \quad (36)$$

Inserting equation (34) in the expression of $\mathbf{E}^{(n)}(\mathbf{r})$ given by equation (24), we obtain

$$\mathbf{E}^{(n)}(\mathbf{r}) = \frac{(-1)^{n-1}}{4\pi\varepsilon_0 n!} \mathcal{P}^{(n)} \|\nabla^{n+1} \frac{1}{r} + \frac{(-1)^{n-1}}{4\pi\varepsilon_0 n!} \mathbf{e}_i \delta_{\{i_1 i_2 \dots i_n\}} \Lambda_{i_3 \dots i_n} \partial_i \partial_{i_1} \dots \partial_{i_n} \frac{1}{r} .$$

and, since the contraction with the fully symmetric tensor $\mathbf{\Lambda}^{(n-2)}$ produces $C_n^2 = n(n-1)/2$ identical terms,

$$\begin{aligned} \mathbf{E}^{(n)}(\mathbf{r}) &= \frac{(-1)^{n-1}}{4\pi\varepsilon_0 n!} \mathcal{P}^{(n)} \|\nabla^{n+1} \frac{1}{r} + \frac{(-1)^{n-1}}{8\pi\varepsilon_0 (n-2)!} \mathbf{e}_i \Lambda_{i_1 \dots i_{n-2}} \partial_i \partial_{i_1} \dots \partial_{i_{n-2}} \Delta \frac{1}{r} \\ &= \frac{(-1)^{n-1}}{4\pi\varepsilon_0 n!} \mathcal{P}^{(n)} \|\nabla^{n+1} \frac{1}{r} + \frac{(-1)^n}{2(n-2)! \varepsilon_0} \mathbf{\Lambda}^{(n-2)} \|\nabla^{n-1} \delta(\mathbf{r}) . \end{aligned} \quad (37)$$

Searching the extension of $\mathbf{E}^{(n)}$ to the entire space, for establishing the δ -type singularities, we have to calculate the limit

$$\begin{aligned} \langle \mathbf{E}_{(0)}^{(n)}, \phi \rangle &= \frac{(-1)^{n-1}}{4\pi\varepsilon_0 n!} \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d^3x \left(\mathcal{P}^{(n)} \|\nabla^{n+1} \frac{1}{r} \right) \phi(\mathbf{r}) \\ &\quad + \frac{(-1)^n}{2(n-2)! \varepsilon_0} \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d^3x \mathbf{\Lambda}^{(n-2)} \|\nabla^{n-1} \delta(\mathbf{r}) \phi(\mathbf{r}) , \end{aligned}$$

or

$$\begin{aligned} \langle \mathbf{E}_{(0)}^{(n)}, \phi \rangle &= \frac{(-1)^{n-1}}{4\pi\varepsilon_0 n!} \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d^3x \left(\mathcal{P}^{(n)} \|\nabla^{n+1} \frac{1}{r} \right) \phi(\mathbf{r}) \\ &\quad - \frac{1}{2(n-2)! \varepsilon_0} \mathbf{\Lambda}^{((n-2))} \|\left(\nabla^{n-1} \phi \right)_0 . \end{aligned} \quad (38)$$

As done in Ref. [1], the limit of the remaining integral from equation (38) can be easily expressed for arbitrary n . Applying the Gauss theorem, we can write

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d^3x \left(\mathcal{P}^{(n)} \|\nabla^{n+1} \frac{1}{r} \right) \phi(\mathbf{r}) \\ &= \lim_{\varepsilon \rightarrow 0} \mathbf{e}_i \left[\oint_{\Sigma_\varepsilon} dS \nu_i \left(\mathcal{P}_{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n} \frac{1}{r} \right) \phi(\mathbf{r}) - \int_{\mathcal{D}_\varepsilon} d^3x \left(\mathcal{P}_{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n} \frac{1}{r} \right) \partial_i \phi(\mathbf{r}) \right] . \end{aligned} \quad (39)$$

Introducing the Taylor series for the function $\phi(\mathbf{r})$ in the surface integral from the last equation and since $r = \varepsilon$ on the sphere Σ_ε , we can write

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \mathbf{e}_i \oint_{\Sigma_\varepsilon} dS \nu_i \left(\mathcal{P}_{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n} \frac{1}{r} \right) \phi(\mathbf{r}) \\ &= 4\pi \mathbf{e}_i \lim_{\varepsilon \rightarrow 0} \sum_{\alpha=0}^{\infty} \frac{\varepsilon^{\alpha-n+1}}{\alpha!} \mathcal{P}_{i_1 \dots i_n} \langle \nu_i C_{i_1 \dots i_n}^{(n)} \nu_{i_{n+1}} \dots \nu_{i_{n+\alpha}} \rangle \left(\partial_{i_{n+1}} \dots \partial_{i_{n+\alpha}} \phi \right)_0 . \end{aligned} \quad (40)$$

Let us evaluate the tensorial contraction which is present of the general term in the series from the previous equation:

$$\mathcal{P}_{i_1 \dots i_n} \langle \nu_i C_{i_1 \dots i_n}^{(n)} \nu_{i_{n+1}} \dots \nu_{i_{n+\alpha}} \rangle .$$

From equation (7) we can easily see that all the terms containing at least a symbol $\delta_{i_q i_s}$ with $1 \leq q, s \leq n$ give null results by contraction with the traceless tensor $\mathcal{P}^{(n)}$. Only the term corresponding to $k = 0$ in equation (7) can give results different from zero for

this contraction. Moreover, a result different from zero can be obtained if and only if $\alpha + 1 \geq n$. Therefore, equation (42) can be written as

$$\lim_{\varepsilon \rightarrow 0} \mathbf{e}_i \oint_{\Sigma_\varepsilon} dS \nu_i \left(\mathcal{P}_{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n} \frac{1}{r} \right) \phi(\mathbf{r}) = 4\pi \mathbf{e}_i (-1)^n (2n-1)!! \\ \times \lim_{\varepsilon \rightarrow 0} \sum_{\alpha=n-1}^{\infty} \frac{\varepsilon^{\alpha-n+1}}{\alpha!} \mathcal{P}_{i_1 \dots i_n} \left\langle \nu_{i_1} \dots \nu_{i_n} \nu_{i_{n+1}} \dots \nu_{i_{n+\alpha}} \nu_i \right\rangle \left(\partial_{i_{n+1}} \dots \partial_{i_{n+\alpha}} \phi \right)_0. \quad (41)$$

For $\alpha > n-1$, the corresponding terms from the series in equation (41) contain positive powers of ε . Consequently, the corresponding limits for $\varepsilon \rightarrow 0$ vanish. From this series only the term for which

$$\alpha = n-1 \quad (42)$$

can be different from zero and the result of the limit in equation (41) is given by

$$\lim_{\varepsilon \rightarrow 0} \mathbf{e}_i \oint_{\Sigma_\varepsilon} dS \nu_i \left(\mathcal{P}_{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n} \frac{1}{r} \right) \phi(\mathbf{r}) = \frac{4\pi (-1)^n (2n-1)!!}{(n-1)!} \\ \times \mathbf{e}_i \mathcal{P}_{i_1 \dots i_n} \left\langle \nu_{i_1} \dots \nu_{i_n} \nu_{i_{n+1}} \dots \nu_{i_{2n-1}} \nu_i \right\rangle \left(\partial_{i_{n+1}} \dots \partial_{i_{2n-1}} \phi \right)_0 \\ = \frac{4\pi (-1)^n (2n-1)!!}{(n-1)!} \left\langle \mathcal{P}^{(n)} || \boldsymbol{\nu}^{2n} \right\rangle || \left(\nabla^{n-1} \phi \right)_0. \quad (43)$$

Let us evaluate now the volume integral in equation (39):

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d^3x \left(\nabla^n || \frac{\mathcal{P}^{(n)}}{r} \right) \nabla \phi(\mathbf{r}) \\ = \lim_{\varepsilon \rightarrow 0} \left[\oint_{\Sigma_\varepsilon} dS \boldsymbol{\nu} || \left(\nabla^{n-1} || \frac{\mathcal{P}^{(n)}}{r} \right) \nabla \phi(\mathbf{r}) - \int_{\mathcal{D}_\varepsilon} d^3x \left(\nabla^{n-1} || \frac{\mathcal{P}^{(n)}}{r} \right) \nabla^2 \phi(\mathbf{r}) \right]. \quad (44)$$

The part corresponding to the surface integral can be written as

$$\lim_{\varepsilon \rightarrow 0} \oint_{\Sigma_\varepsilon} dS \boldsymbol{\nu} || \left(\nabla^{n-1} || \frac{\mathcal{P}^{(n)}}{r} \right) \nabla \phi(\mathbf{r}) = \mathbf{e}_i \mathcal{P}_{i_1 \dots i_n} \lim_{\varepsilon \rightarrow 0} \oint_{\Sigma_\varepsilon} dS \nu_{i_n} \left(\partial_{i_1} \dots \partial_{i_{n-1}} \frac{1}{r} \right) \partial_i \phi(\mathbf{r}). \quad (45)$$

Introducing the Taylor series for $\phi(\mathbf{r})$ and standing out the average over $\boldsymbol{\nu}$, we obtain

$$\lim_{\varepsilon \rightarrow 0} \oint_{\Sigma_\varepsilon} dS \boldsymbol{\nu} || \left(\nabla^{n-1} || \frac{\mathcal{P}^{(n)}}{r} \right) \nabla \phi(\mathbf{r}) \\ = 4\pi \mathbf{e}_i \lim_{\varepsilon \rightarrow 0} \sum_{\alpha=0}^{\infty} \frac{\varepsilon^{\alpha-n+2}}{\alpha!} \mathcal{P}_{i_1 \dots i_n} \left\langle C_{i_1 \dots i_{n-1}}^{(n-1, n-1)} \nu_{i_n} \dots \nu_{i_{n+\alpha}} \right\rangle \times \left(\partial_{i_{n+1}} \dots \partial_{i_{n+\alpha}} \partial_i \phi \right)_0. \quad (46)$$

Analogously to the reasoning from the previous case, we can see that the limit is zero since for $\alpha = n-2$

$$\mathcal{P}_{i_1 \dots i_n} \left\langle C_{i_1 \dots i_{n-1}}^{(n-1, n-1)} \nu_{i_n} \nu_{i_{n+1}} \dots \nu_{i_{2n-2}} \right\rangle = 0.$$

Therefore,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d^3x \left(\nabla^n || \frac{\mathcal{P}^{(n)}}{r} \right) \nabla \phi(\mathbf{r}) = - \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d^3x \left(\nabla^{n-1} || \frac{\mathcal{P}^{(n)}}{r} \right) \nabla^2 \phi(\mathbf{r}), \quad (47)$$

and by repeatedly applying the procedure, all terms cancel. Finally, only the surface integral from equation (39) gives a limit different from zero and

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d^3x \left(\mathcal{P}^{(n)} || \nabla^{n+1} \frac{1}{r} \right) \phi(\mathbf{r}) = \frac{4\pi(-1)^n(2n-1)!!}{(n-1)!} \langle \mathcal{P}^{(n)} || \nu^{2n} \rangle || (\nabla^{n-1} \phi)_0 . \quad (48)$$

Let us consider the contraction

$$\langle \mathcal{P}^{(n)} || \nu^{2n} \rangle = \mathcal{P}_{i_1 \dots i_n} \langle \nu_{i_1} \dots \nu_{i_n} \nu_{j_1} \dots \nu_{j_n} \rangle .$$

To this contraction contribute only the terms from the average of the ν -product not containing factors $\delta_{i_k i_l}$, with $1 \leq k, l \leq n$. According to equation (15), the terms giving non-zero contributions are of the form

$$\frac{1}{(2n+1)!!} \delta_{i_1 j_1} \dots \delta_{i_n j_n}$$

and all such terms are obtained considering the $n!$ permutations of the indices $j_1 \dots j_n$ in this product. Therefore, the final expression in equation (43) is given by

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d^3x \left(\mathcal{P}^{(n)} || \nabla^{n+1} \frac{1}{r} \right) \phi(\mathbf{r}) = \frac{4\pi(-1)^n n}{2n+1} \mathcal{P}^{(n)} || (\nabla^{n-1} \phi)_0 . \quad (49)$$

Equation (38), with equation (49) inserted in, yields

$$\langle \mathbf{E}_{(0)}^{(n)}, \phi \rangle = -\frac{1}{(n-1)!(2n+1)\varepsilon_0} \mathcal{P}^{(n)} || (\nabla^{n-1} \phi)_0 - \frac{1}{2(n-2)!\varepsilon_0} \mathbf{\Lambda}^{(n-2)} || (\nabla^{n-1} \phi)_0 . \quad (50)$$

Therefore, the delta-form distribution associated to the 2^n -polar electric field is given by

$$\mathbf{E}_{(0)}^{(n)} = \frac{(-1)^n}{(n-1)!(2n+1)\varepsilon_0} \mathcal{P}^{(n)} || \nabla^{n-1} \delta(\mathbf{r}) + \frac{(-1)^n}{2(n-2)!\varepsilon_0} \mathbf{\Lambda}^{(n-2)} || \nabla^{n-1} \delta(\mathbf{r}) . \quad (51)$$

One can easily see that the result (32) for $n = 2$ is, indeed, a particular case of the formula (51). In the case $n = 3$, equation (51) becomes:

$$\mathbf{E}_{(0)}^{(3)} = -\frac{1}{14\varepsilon_0} \mathcal{P}^{(3)} || \nabla^2 \delta(\mathbf{r}) - \frac{1}{2\varepsilon_0} \mathbf{\Lambda}^{(1)} || \nabla^2 \delta(\mathbf{r}) .$$

The result can be expressed in terms of primitive tensors $\mathbf{P}^{(n)}$ introducing equation (34) in equation (51):

$$\begin{aligned} \mathbf{E}_{(0)}^{(n)} &= \frac{(-1)^n}{(n-1)!(2n+1)\varepsilon_0} \mathbf{P}^{(n)} || \nabla^{n-1} \delta(\mathbf{r}) + \frac{(-1)^n}{2(n-2)!\varepsilon_0} \mathbf{\Lambda}^{(n-2)} || \nabla^{n-1} \delta(\mathbf{r}) \\ &\quad - \frac{(-1)^n}{(n-1)!(2n+1)\varepsilon_0} \mathbf{e}_i \delta_{\{i i_1 i_2 \dots i_{n-1}\}} \partial_{i_1} \dots \partial_{i_{n-1}} \delta(\mathbf{r}) . \end{aligned}$$

In the last term from this equation, there are $n-1$ identical terms of the form

$$\mathbf{e}_i \mathbf{\Lambda}_{i_2 \dots i_{n-1}} \partial_i \partial_{i_2} \dots \partial_{i_{n-1}} \delta(\mathbf{r}) = \mathbf{\Lambda}^{(n-2)} || \nabla^{n-1} \delta(\mathbf{r})$$

and $C_{n-1}^2 = (n-1)(n-2)/2$ terms of the form

$$\mathbf{e}_i \mathbf{\Lambda}_{i i_3 \dots i_{n-1}} \partial_{i_3} \dots \partial_{i_{n-1}} \delta(\mathbf{r}) = \mathbf{\Lambda}^{(n-2)} || \nabla^{n-3} \Delta \delta(\mathbf{r}) .$$

Finally,

$$\begin{aligned} \mathbf{E}_{(0)}^{(n)} = & \frac{(-1)^n}{(n-1)!(2n+1)\varepsilon_0} \mathbf{P}^{(n)} \|\nabla^{n-1} \delta(\mathbf{r}) + \frac{(-1)^n(2n-1)}{2(n-2)!(2n+1)\varepsilon_0} \mathbf{A}^{(n-2)} \|\nabla^{n-1} \delta(\mathbf{r}) \\ & - \frac{(-1)^n}{2(n-3)!(2n+1)\varepsilon_0} \mathbf{A}^{(n-2)} \|\nabla^{n-3} \Delta \delta(\mathbf{r}) . \end{aligned} \quad (52)$$

This last equation becomes equation (29) in case $n = 2$. The terms containing $\Delta \delta(\mathbf{r})$ are present beginning from $n = 3$.

4. Singularities of the magnetostatic field

For the vector potential in the exterior of the domain \mathcal{D} , we have

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla \times \left(\nabla^{n-1} \|\mathbf{M}^{(n)}\| \right) \\ &= \frac{\mu_0}{4\pi} \mathbf{e}_i \varepsilon_{ijk} \partial_j \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \partial_{i_1} \dots \partial_{i_{n-1}} \frac{\mathbf{M}_{i_1 \dots i_{n-1} k}}{r}, \end{aligned} \quad (53)$$

where $\mathbf{M}^{(n)}$ is the magnetic n -th order moment defined by the Cartesian components [13]:

$$\mathbf{M}_{i_1 \dots i_n}(t) = \frac{n}{n+1} \int_{\mathcal{D}} d^3x \, x_{i_1} \dots x_{i_{n-1}} (\mathbf{r} \times \mathbf{J}(\mathbf{r}, t))_{i_n}, \quad (54)$$

or, with tensorial notation:

$$\mathbf{M}^{(n)}(t) = \frac{n}{n+1} \int_{\mathcal{D}} d^3x \, \mathbf{r}^n \times \mathbf{J}(\mathbf{r}, t) .$$

The corresponding expansion of the magnetic field $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$ is given by

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \nabla \times \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla \times \left(\nabla^{n-1} \|\frac{\mathbf{M}^{(n)}}{r}\| \right) \\ &= \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \left[\nabla \cdot \left(\nabla^n \|\frac{\mathbf{M}^{(n)}}{r}\| \right) - \Delta \left(\nabla^{n-1} \|\frac{\mathbf{M}^{(n)}}{r}\| \right) \right] . \end{aligned} \quad (55)$$

For $r \neq 0$, the last term containing $\Delta(1/r)$ is not contributing to the multipole expansion since $\Delta(1/r) = 0$, but, searching the extension of this expansion to the entire space, including the point O , we have to consider it. This term, extended as in the electrostatic case, as $4\pi \mathbf{M}^{(n)} \|\nabla^{n-1} \delta(\mathbf{r})$, is considered as a first extension of \mathbf{B} . It remains to process the limit that implies the expression $\nabla \left(\mathbf{M}^{(n)} \|\nabla^n(1/r)\| \right)$.

In the dipolar case, we write

$$\mathbf{B}^{(1)} = \frac{\mu_0}{4\pi} \mathbf{e}_i \partial_i \partial_j \frac{m_j}{r} + 4\pi \mathbf{m} \delta(\mathbf{r})$$

and, applying equation (16), we obtain for the delta-singularity, the well-known expression [2]:

$$\mathbf{B}_{(0)}^{(1)} = -\frac{\mu_0}{3} \mathbf{m} \delta(\mathbf{r}) + \mu_0 \mathbf{m} \delta(\mathbf{r}) = \frac{2\mu_0}{3} \mathbf{m} \delta(\mathbf{r}) . \quad (56)$$

For the higher multipolar orders, let us begin with the 4-polar term:

$$\mathbf{B}^{(2)} = -\frac{\mu_0}{8\pi} \nabla \left(\nabla^2 \left\| \frac{\mathbf{M}^{(2)}}{r} \right\| \right) = -\frac{\mu_0}{8\pi} \mathbf{e}_i \mathbf{M}_{jk} \partial_i \partial_j \partial_k \frac{1}{r} - \frac{\mu_0}{2} \mathbf{e}_i \mathbf{M}_{ji} \partial_j \delta(\mathbf{r}) .$$

Applying equation (20),

$$\mathbf{B}_{(0)}^{(2)} = \frac{\mu_0}{10} \mathbf{e}_i \mathbf{M}_{jk} \delta_{\{ij} \partial_{k\}} \delta(\mathbf{r}) - \frac{\mu_0}{2} \mathbf{e}_i \mathbf{M}_{ji} \partial_j \delta(\mathbf{r}) = \frac{\mu_0}{10} \mathbf{e}_i (\mathbf{M}_{ij} + \mathbf{M}_{ji}) \partial_j \delta(\mathbf{r}) - \frac{\mu_0}{2} \mathbf{e}_i \mathbf{M}_{ji} \partial_j \delta(\mathbf{r})$$

since $\delta_{jk} \mathbf{M}_{jk} = \mathbf{M}_{jj} = 0$. Further,

$$\mathbf{B}_{(0)}^{(2)} = \frac{\mu_0}{5} \mathbf{e}_i \overset{\leftrightarrow}{\mathbf{M}}_{ij} \partial_j \delta(\mathbf{r}) - \frac{\mu_0}{2} \mathbf{e}_i \mathbf{M}_{ji} \partial_j \delta(\mathbf{r}) = \frac{\mu_0}{5} \overset{\leftrightarrow}{\mathbf{M}}^{(2)} \|\nabla \delta(\mathbf{r}) - \frac{\mu_0}{2} (\nabla \delta(\mathbf{r})) \|\mathbf{M}^{(2)} , \quad (57)$$

where $\overset{\leftrightarrow}{\mathbf{M}}^{(2)}$ is the symmetric part of the tensor $\mathbf{M}^{(2)}$ corresponding to the identity

$$\mathbf{M}_{ij} = \frac{1}{2} (\mathbf{M}_{ij} + \mathbf{M}_{ji}) + \frac{1}{2} (\mathbf{M}_{ij} - \mathbf{M}_{ji}) = \overset{\leftrightarrow}{\mathbf{M}}_{ij} + \frac{1}{2} \varepsilon_{ijk} \mathbf{N}_k . \quad (58)$$

The antisymmetric part is expressed in terms of the components of a first rank tensor:

$$\mathbf{N}_i = \varepsilon_{ijk} \mathbf{M}_{jk} = \frac{2}{3} \int_{\mathcal{D}} d^3x [\mathbf{r} \times (\mathbf{r} \times \mathbf{J})]_i .$$

For $n = 2$, the symmetric projection of the magnetic moment is an **STF** tensor, i.e, denoting by $\mathcal{M}^{(n)}$ the **STF** magnetic moments,

$$\mathcal{M}^{(2)} = \overset{\leftrightarrow}{\mathbf{M}}^{(2)} ,$$

the equation (58) is written as

$$\mathbf{M}_{ij} = \mathcal{M}_{ij} + \frac{1}{2} \varepsilon_{ijk} \mathbf{N}_k . \quad (59)$$

The introduction of these result in equation (57) gives

$$\mathbf{B}_{(0)}^{(2)} = -\frac{3\mu_0}{10} \mathcal{M}^{(2)} \|\nabla \delta(\mathbf{r}) - \frac{\mu_0}{4} \mathbf{N} \times \nabla \delta(\mathbf{r}) , \quad (60)$$

where $\mathbf{N} = \mathbf{N}_i \mathbf{e}_i$.

Beginning from $n = 3$, the symmetric part of the magnetic moment tensor is not the same with the **STF** one. We write the identity

$$\mathbf{M}_{i_1 i_2 i_3} = \frac{1}{3} (\mathbf{M}_{i_1 i_2 i_3} + \mathbf{M}_{i_2 i_3 i_1} + \mathbf{M}_{i_1 i_3 i_2}) + \frac{1}{3} [(\mathbf{M}_{i_1 i_2 i_3} - \mathbf{M}_{i_2 i_3 i_1}) + (\mathbf{M}_{i_1 i_2 i_3} - \mathbf{M}_{i_1 i_3 i_2})] , \quad (61)$$

where $\mathbf{M}_{i_1 i_2 i_3}$ is symmetric in the first two indices. The first parenthesis represents the fully symmetric part of the tensor $\mathbf{M}^{(3)}$ and the second one can be expressed in terms of the second order tensor $\mathbf{N}^{(2)}$ defined by the components

$$\mathbf{N}_{i_1 i_2} = \varepsilon_{i_2 p q} \mathbf{M}_{i_1 p q} = \frac{3}{4} \int_{\mathcal{D}} d^3x x_{i_1} [\mathbf{r} \times (\mathbf{r} \times \mathbf{J})]_{i_2} ,$$

with the relationships:

$$\mathbf{M}_{i_1 i_2 i_3} - \mathbf{M}_{i_2 i_3 i_1} = \varepsilon_{i_1 i_3 q} \mathbf{N}_{i_2 q} , \quad \mathbf{M}_{i_1 i_2 i_3} - \mathbf{M}_{i_1 i_3 i_2} = \varepsilon_{i_2 i_3 q} \mathbf{N}_{i_1 q} .$$

We can write

$$(\mathbf{M}_{i_1 i_2 i_3} - \mathbf{M}_{i_2 i_3 i_1}) + (\mathbf{M}_{i_1 i_2 i_3} - \mathbf{M}_{i_1 i_3 i_2}) = \varepsilon_{i_1 i_3 q} \mathbf{N}_{i_2 q} + \varepsilon_{i_2 i_3 q} \mathbf{N}_{i_1 q} = \sum_{\lambda=1}^2 \varepsilon_{i_{\lambda} i_3 q} \mathbf{N}_{(i_1 i_2) q}^{(\lambda)} ,$$

where by the notation $\mathbf{N}_{(i_1 i_2 q)}^{(\lambda)}$ we understand the component without the index i_λ . Employing the above definitions and notation, equation (61) can be written as

$$\mathbf{M}_{i_1 i_2 i_3} = \overset{\leftrightarrow}{\mathbf{M}}_{i_1 i_2 i_3} + \frac{1}{3} \sum_{\lambda=1}^2 \varepsilon_{i_\lambda i_3 q} \mathbf{N}_{(i_1 i_2 q)}^{(\lambda)}, \quad (62)$$

where $\overset{\leftrightarrow}{\mathbf{M}}^{(3)}$ is the symmetric part of the tensor $\mathbf{M}^{(3)}$.

Let us write the third order term from the magnetic field expansion (55)

$$\mathbf{B}^{(3)}(\mathbf{r}) = \frac{\mu_0}{24\pi} \nabla \left(\nabla^3 \parallel \frac{\mathbf{M}^{(3)}}{r} \right) - \frac{\mu_0}{24\pi} \nabla^2 \parallel \left(\Delta \frac{\mathbf{M}^{(3)}}{r} \right).$$

Retaining the second expression which represent an extension as distribution with the point-like support O :

$$\mathbf{B}^{(3)}(\mathbf{r}) = \frac{\mu_0}{24\pi} \nabla \left(\nabla^3 \parallel \frac{\mathbf{M}^{(3)}}{r} \right) + \frac{\mu_0}{6} (\nabla^2 \delta(\mathbf{r})) \parallel \mathbf{M}^{(3)}. \quad (63)$$

Introducing the symmetric tensor $\overset{\leftrightarrow}{\mathbf{M}}^{(3)}$ from equation (62) in the first expression of the right-hand side of the above equation,

$$\nabla \left(\nabla^3 \parallel \frac{\mathbf{M}^{(3)}}{r} \right) = \nabla \left(\nabla^3 \parallel \frac{\overset{\leftrightarrow}{\mathbf{M}}^{(3)}}{r} \right) + \frac{1}{3} \mathbf{e}_i \partial_i \partial_{i_1} \partial_{i_2} \partial_{i_3} \frac{1}{r} \sum_{\lambda=1}^2 \varepsilon_{i_\lambda i_3 q} \mathbf{N}_{(i_1 i_2 q)}^{(\lambda)}.$$

Since in the last expression, the two terms of the sum contain either the $\partial_{i_1} \partial_{i_3} \varepsilon_{i_1 i_3 q}$ or $\partial_{i_2} \partial_{i_3} \varepsilon_{i_2 i_3 q}$ which vanish, we can write

$$\nabla \left(\nabla^3 \parallel \frac{\mathbf{M}^{(3)}}{r} \right) = \nabla \left(\nabla^3 \parallel \frac{\overset{\leftrightarrow}{\mathbf{M}}^{(3)}}{r} \right). \quad (64)$$

This result expresses the invariance of the multipole expansion of the magnetic field to the substitution $\mathbf{M}^{(3)} \rightarrow \overset{\leftrightarrow}{\mathbf{M}}^{(3)}$. The introduction of $\overset{\leftrightarrow}{\mathbf{M}}^{(3)}$ in the δ -type singularity from equation (63) gives

$$\begin{aligned} (\nabla^2 \delta(\mathbf{r})) \parallel \mathbf{M}^{(3)} &= (\nabla^2 \delta(\mathbf{r})) \parallel \overset{\leftrightarrow}{\mathbf{M}}^{(3)} + \frac{1}{3} \mathbf{e}_i \partial_{i_1} \partial_{i_2} \delta(\mathbf{r}) \sum_{\lambda=1}^2 \varepsilon_{i_\lambda i q} \mathbf{N}_{(i_1 i_2 q)}^{(\lambda)} \\ &= (\nabla^2 \delta(\mathbf{r})) \parallel \overset{\leftrightarrow}{\mathbf{M}}^{(3)} + \frac{2}{3} \mathbf{e}_i \mathbf{N}_{i_2 q} \varepsilon_{i q i_1} \partial_{i_1} \partial_{i_2} \delta(\mathbf{r}) = \overset{\leftrightarrow}{\mathbf{M}}^{(3)} \parallel \nabla^2 \delta(\mathbf{r}) + \frac{2}{3} \mathbf{e}_i \mathbf{N}_{i_2 q} \varepsilon_{i q i_1} \partial_{i_1} \partial_{i_2} \delta(\mathbf{r}) \end{aligned}$$

since $\overset{\leftrightarrow}{\mathbf{M}}$ is symmetric. For expressing in a compact form such tensorial contraction, let us introduce the notation

$$\mathbf{A}^{(n)} \mid \times \mid \mathbf{B}^{(n)} = \mathbf{e}_i \mathbf{A}_{i_1 \dots i_{n-1} q} \varepsilon_{i q s} \mathbf{B}_{s i_1 \dots i_{n-1}}. \quad (65)$$

Then,

$$(\nabla^2 \delta(\mathbf{r})) \parallel \mathbf{M}^{(3)} = \overset{\leftrightarrow}{\mathbf{M}}^{(3)} \parallel \nabla^2 \delta(\mathbf{r}) + \frac{2}{3} \mathbf{N}^{(2)} \mid \times \mid \nabla^2 \delta(\mathbf{r}). \quad (66)$$

The octupolar term $\mathbf{B}^{(3)}$ including partially δ -form singularities becomes

$$\mathbf{B}^{(3)}(\mathbf{r}) = \frac{\mu_0}{24\pi} \nabla \left(\nabla^3 \left\| \frac{\overset{\leftrightarrow}{\mathbf{M}}^{(3)}}{r} \right\| \right) + \frac{\mu_0}{6} \overset{\leftrightarrow}{\mathbf{M}}^{(3)} \left\| \nabla^2 \delta(\mathbf{r}) + \frac{\mu_0}{9} \mathbf{N}^{(2)} \right\| \times \left\| \nabla^2 \delta(\mathbf{r}) \right\|. \quad (67)$$

The **STF** projection $\mathcal{M}^{(3)}$, up to a numerical factor, is given by

$$\mathcal{M}_{i_1 i_2 i_3} = \overset{\leftrightarrow}{\mathbf{M}}_{i_1 i_2 i_3} - \delta_{\{i_1 i_2\}} \tilde{\Lambda}_{i_3}(\overset{\leftrightarrow}{\mathbf{M}}^{(3)}) , \quad (68)$$

where the symmetric tensor $\tilde{\Lambda}^{(n-2)}$ corresponds to $\mathbf{M}^{(n)}$ by the a formula of the type (34). It easy to see that

$$\tilde{\Lambda}_i = \frac{1}{5} \overset{\leftrightarrow}{\mathbf{M}}_{iqq} = \frac{1}{15} \mathbf{M}_{qqi} = \frac{1}{20} \int_{\mathcal{D}} d^3x r^2 (\mathbf{r} \times \mathbf{J})_i . \quad (69)$$

The introduction of equation (68) in equation (67) gives

$$\begin{aligned} \mathbf{B}^{(3)}(\mathbf{r}) &= \frac{\mu_0}{24\pi} \nabla \left(\nabla^3 \left\| \frac{\mathcal{M}^{(3)}}{r} \right\| \right) + \frac{\mu_0}{6} \mathcal{M}^{(3)} \left\| \nabla^2 \delta(\mathbf{r}) - \frac{\mu_0}{6} \tilde{\Lambda} \left\| \nabla^2 \delta(\mathbf{r}) + \frac{\mu_0}{6} \tilde{\Lambda} \Delta \delta(\mathbf{r}) \right\| \right. \\ &\quad \left. + \frac{\mu_0}{9} \mathbf{N}^{(2)} \right\| \times \left\| \nabla^2 \delta(\mathbf{r}) \right\|. \end{aligned} \quad (70)$$

From this last equation, it is seen that the multipole expansion of $\mathbf{B}(\mathbf{r})$ ($r \neq 0$) is invariant to the substitution $\mathbf{M}^{(3)} \rightarrow \mathcal{M}^{(3)}$ [9, 10].

It remains to calculate the extension of the first term from equation (70) to the entire space. For this, we have the result (49) from the case of electrostatic field which, for arbitrary n , gives in the present case:

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{D}_\varepsilon} d^3x \nabla \left(\nabla^n \left\| \frac{\mathcal{M}^{(n)}}{r} \right\| \right) \phi(\mathbf{r}) = \frac{4\pi(-1)^n n}{2n+1} \mathcal{M}^{(n)} \left\| \left(\nabla^{n-1} \phi \right)_0 \right\|. \quad (71)$$

The final result for the singular part of $\mathbf{B}^{(3)}$ having as support the point O is:

$$\mathbf{B}_{(0)}^{(3)} = \frac{2\mu_0}{21} \mathcal{M}^{(3)} \left\| \nabla^2 \delta(\mathbf{r}) - \frac{\mu_0}{6} \tilde{\Lambda} \left\| \nabla^2 \delta(\mathbf{r}) + \frac{\mu_0}{6} \tilde{\Lambda} \Delta \delta(\mathbf{r}) + \frac{\mu_0}{9} \mathbf{N}^{(2)} \right\| \times \left\| \nabla^2 \delta(\mathbf{r}) \right\| \right\|. \quad (72)$$

Let us consider the extension for an arbitrary n of

$$\mathbf{B}^{(n)}(\mathbf{r}) = \frac{\mu_0(-1)^{n-1}}{4\pi n!} \nabla \left(\nabla^n \left\| \frac{\mathbf{M}^{(n)}}{r} \right\| \right) + \frac{\mu_0(-1)^{n-1}}{n!} \left(\nabla^{n-1} \delta(\mathbf{r}) \right) \left\| \mathbf{M}^{(n)} \right\|. \quad (73)$$

The generalized equation (62) is given by

$$\mathbf{M}_{i_1 \dots i_n} = \overset{\leftrightarrow}{\mathbf{M}}_{i_1 \dots i_n} + \frac{1}{n} \sum_{\lambda=1}^{n-1} \varepsilon_{i_\lambda i_n q} \mathbf{N}_{(i_1 \dots i_{n-1} q)}^{(\lambda)} , \quad (74)$$

where it is introduced the $(n-1)$ -th order tensor $\mathbf{N}^{(n-1)}$, partial symmetric in the first $n-2$ indices and with null contraction of the last index i_{n-1} with any of the indices i_q , $q < n-1$:

$$\mathbf{N}_{i_1 \dots i_{n-1}} = \varepsilon_{i_{n-1} p s} \mathbf{M}_{i_1 \dots i_{n-2} p s} = \frac{n}{n+1} \int_{\mathcal{D}} d^3x x_{i_1} \dots x_{i_{n-2}} [\mathbf{r} \times (\mathbf{r} \times \mathbf{J})]_{i_{n-1}} . \quad (75)$$

Inserting equation (74) in the expression of $\mathbf{B}^{(n)}(\mathbf{r})$, we consider the different terms from equation (73):

$$\begin{aligned}\nabla \left(\nabla^n \parallel \frac{\mathbf{M}^{(n)}}{r} \right) &= \nabla \left(\nabla^n \parallel \frac{\overset{\leftrightarrow}{\mathbf{M}}^{(n)}}{r} \right) + \frac{1}{n} \mathbf{e}_i \left(\partial_i \partial_{i_1} \dots \partial_{i_n} \frac{1}{r} \right) \sum_{\lambda=1}^{n-1} \varepsilon_{i_\lambda i_n q} \mathbf{N}_{(i_1 \dots i_{n-1} q)}^{(\lambda)} \\ &= \nabla \left(\nabla^n \parallel \frac{\overset{\leftrightarrow}{\mathbf{M}}^{(n)}}{r} \right)\end{aligned}$$

since all the terms from the last sum contain a contraction of the type $\varepsilon_{i_\lambda i_n q} \partial_{i_\lambda} \partial_{i_n}$ with $1 \leq l \leq n-1$.

$$\begin{aligned}(\nabla^{n-1} \delta(\mathbf{r})) \parallel \mathbf{M}^{(n)} &= (\nabla^{n-1} \delta(\mathbf{r})) \parallel \overset{\leftrightarrow}{\mathbf{M}}^{(n)} + \mathbf{e}_i \frac{1}{n} \left(\partial_{i_1} \dots \partial_{i_{n-1}} \delta(\mathbf{r}) \right) \sum_{\lambda=1}^{n-1} \varepsilon_{i_\lambda i_n q} \mathbf{N}_{(i_1 \dots i_{n-1} q)}^{(\lambda)} \\ &= (\nabla^{n-1} \delta(\mathbf{r})) \parallel \overset{\leftrightarrow}{\mathbf{M}}^{(n)} + \frac{n-1}{n} \mathbf{e}_i \mathbf{N}_{i_2 \dots i_{n-1} q} \varepsilon_{iq i_1} \partial_{i_1} \partial_{i_2} \dots \partial_{i_{n-1}} \delta(\mathbf{r}) \\ &= \overset{\leftrightarrow}{\mathbf{M}}^{(n)} \parallel \nabla^{n-1} \delta(\mathbf{r}) + \frac{n-1}{n} \mathbf{N}^{(n-1)} \parallel \nabla^{n-1} \delta(\mathbf{r}),\end{aligned}$$

using the notation (65). The final result for the substitution of equation (74) in equation (73) can be written as

$$\begin{aligned}\mathbf{B}^{(n)}(\mathbf{r}) &= \frac{\mu_0 (-1)^{n-1}}{4\pi n!} \nabla \left(\nabla^n \parallel \frac{\overset{\leftrightarrow}{\mathbf{M}}^{(n)}}{r} \right) + \frac{\mu_0 (-1)^{n-1}}{n!} \overset{\leftrightarrow}{\mathbf{M}}^{(n)} \parallel \nabla^{n-1} \delta(\mathbf{r}) \\ &\quad + \frac{\mu_0 (-1)^{n-1} (n-1)}{n! n} \mathbf{N}^{(n-1)} \parallel \nabla^{n-1} \delta(\mathbf{r}).\end{aligned}\tag{76}$$

Introducing the **STF** tensor $\mathcal{M}^{(n)}$ with the components given by an equation of the type (34)

$$\overset{\leftrightarrow}{\mathbf{M}}_{i_1 \dots i_n} = \mathcal{M}_{i_1 \dots i_n} + \delta_{\{i_1 i_2} \tilde{\Lambda}_{i_3 \dots i_n\}}$$

and considering the different terms from equation (76), we obtain

$$\begin{aligned}\nabla \left(\nabla^n \parallel \frac{\overset{\leftrightarrow}{\mathbf{M}}^{(n)}}{r} \right) &= \nabla \left(\nabla^n \parallel \frac{\mathcal{M}^{(n)}}{r} \right) + \mathbf{e}_i \partial_i \left(\partial_{i_1} \dots \partial_{i_n} \frac{1}{r} \right) \delta_{\{i_1 i_2} \tilde{\Lambda}_{i_3 \dots i_n\}} \\ &= \nabla \left(\nabla^n \parallel \frac{\mathcal{M}^{(n)}}{r} \right) + \frac{n(n-1)}{2} \mathbf{e}_i \left(\partial_{i_3} \dots \partial_{i_n} \Delta \frac{1}{r} \right) \tilde{\Lambda}_{i_3 \dots i_n} \\ &= \nabla \left(\nabla^n \parallel \frac{\mathcal{M}^{(n)}}{r} \right) - 2\pi n(n-1) \tilde{\Lambda}^{(n-2)} \parallel \nabla^{n-1} \delta(\mathbf{r}).\end{aligned}$$

The second tensorial contraction from equation (76) can be written as

$$\overset{\leftrightarrow}{\mathbf{M}}^{(n)} \parallel \nabla^{n-1} \delta(\mathbf{r}) = \mathcal{M}^{(n)} \parallel \nabla^{n-1} \delta(\mathbf{r}) + \mathbf{e}_{i_n} \partial_{i_1} \dots \partial_{i_{n-1}} \delta(\mathbf{r}) \delta_{\{i_1 i_2} \tilde{\Lambda}_{i_3 \dots i_n\}}.$$

In the last expression, there are $n-1$ terms containing the factor $\delta_{i_q i_n}$, $q = 1, \dots, n-1$ and $C_{n-1}^2 = (n-1)(n-2)/2$ terms containing the factor $\delta_{i_q i_p}$ with q and s between 1

and $n - 1$, such that the last equation can be written as

$$\begin{aligned} \overset{\leftrightarrow}{\mathbf{M}}^{(n)} || \nabla^{n-1} \delta(\mathbf{r}) &= \mathcal{M}^{(n)} || \nabla^{n-1} \delta(\mathbf{r}) + (n-1) \tilde{\mathbf{\Lambda}}^{(n-2)} || \nabla^{n-1} \delta(\mathbf{r}) \\ &+ \frac{(n-1)(n-2)}{2} \tilde{\mathbf{\Lambda}}^{(n-2)} || \nabla^{n-3} \Delta \delta(\mathbf{r}) . \end{aligned} \quad (77)$$

Collecting all the above results, equation (76) can be written as

$$\begin{aligned} \mathbf{B}^{(n)}(\mathbf{r}) &= \frac{(-1)^{n-1} \mu_0}{n!} \left[\frac{1}{4\pi} \nabla^{n+1} || \frac{\mathcal{M}^{(n)}}{r} + \mathcal{M}^{(n)} || \nabla^{n-1} \delta(\mathbf{r}) \right. \\ &- \frac{(n-1)(n-2)}{2} \tilde{\mathbf{\Lambda}}^{(n-2)} || \nabla^{n-1} \delta(\mathbf{r}) + \frac{(n-1)(n-2)}{2} \tilde{\mathbf{\Lambda}}^{(n-2)} || \nabla^{(n-3)} \Delta \delta(\mathbf{r}) \\ &\left. + \frac{n-1}{n} \mathbf{N}^{(n-1)} | \times | \nabla^{n-1} \delta(\mathbf{r}) \right] . \end{aligned} \quad (78)$$

Employing equation (71), we finally write the singular δ -form part of $\mathbf{B}^{(n)}$:

$$\begin{aligned} \mathbf{B}^{(n)}(\mathbf{r}) &= \frac{(-1)^{n-1} \mu_0}{n!} \left[-\frac{n}{2n+1} \mathcal{M}^{(n)} || \nabla^{n-1} \delta(\mathbf{r}) - \frac{(n-1)(n-2)}{2} \tilde{\mathbf{\Lambda}}^{(n-2)} || \nabla^{n-1} \delta(\mathbf{r}) \right. \\ &\left. + \frac{(n-1)(n-2)}{2} \tilde{\mathbf{\Lambda}}^{(n-2)} || \nabla^{(n-3)} \Delta \delta(\mathbf{r}) + \frac{n-1}{n} \mathbf{N}^{(n-1)} | \times | \nabla^{n-1} \delta(\mathbf{r}) \right] . \end{aligned} \quad (79)$$

One can easily verify that equations (60) and (72) are particular cases of equation (79).

5. Conclusion

The results of the present paper concerning the δ -form singularities of the electromagnetic field in the static cases are not so appealing as the ones done in Ref. [1]. However, in [1] the corresponding results are obtained, in our opinion, without employing the full content of the multipole expansions. The unpleasant presence of the parameters $\mathbf{\Lambda}$ and \mathbf{N} in the expressions of the δ -type singularities is a consequence of the hypothesis that the basic multipolar moments are the primitive ones ($\mathbf{P}^{(n)}$ in the electric case and $\mathbf{M}^{(n)}$ in the magnetic one). The employment of irreducible representations by the **STF** tensors has the advantage of simplicity in expressing some quantities in several circumstances. It seems, for example, that some trouble appears when employing the multipole expansions in spherical coordinates. Let us consider the example of the electrostatic potential multipole expansion:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{Q_{lm}}{r^{l+1}} Y_{lm}(\theta, \varphi)$$

and the particular case $l = 2$:

$$\Phi^{(2)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \sum_{m=-2}^2 Q_{2m} Y_{2m} . \quad (80)$$

The spherical moments Q_{2m} are linear combinations of the components \mathcal{P}_{ij} of the **STF** moment $\mathcal{P}^{(2)}$. Writing this term in Cartesian coordinates,

$$\Phi^{(2)}(\mathbf{r}) = \frac{1}{8\pi\epsilon_0} \mathbf{P}^{(2)} || \nabla^2 \frac{1}{r} = \frac{1}{8\pi\epsilon_0} \mathcal{P}_{ij} \partial_i \partial_j \frac{1}{r} ,$$

and introducing the **STF** tensor $\mathcal{P}^{(2)}$,

$$\Phi^{(2)}(\mathbf{r}) = \frac{1}{8\pi\epsilon_0} \left[\mathcal{P}_{ij} \partial_i \partial_j \frac{1}{r} + \Lambda \Delta \frac{1}{r} \right] ,$$

i.e

$$\Phi^{(2)}(\mathbf{r}) = \frac{1}{8\pi\epsilon_0} \mathcal{P}_{ij} \partial_i \partial_j \frac{1}{r} - \frac{1}{2\epsilon_0} \Lambda \delta(\mathbf{r}) .$$

For the first term, equation (16) gives the δ -singularity

$$-\frac{1}{6\epsilon_0} \mathcal{P}_{ij} \delta_{ij} \delta(\mathbf{r}) = 0 ,$$

since $\mathcal{P}_{ii} = 0$. Therefore, the electric dipolar potential $\Phi^{(2)}$ has a delta-type singularity

$$\left(\Phi^{(2)} \right)_{(0)} = -\frac{1}{2\epsilon_0} \Lambda \delta(\mathbf{r}). \quad (81)$$

But, for the term expressed in spherical coordinates (equation (80)), since there are no derivatives of $1/r$, one has no δ -type singularities.

In our opinion, equation (81) represents the correct result. From the given example we also see that some care is necessary when the invariance properties of the multipole expansions are used.

If correct, the results of the present paper can be useful in classical and quantum physics, in the second case starting with the problem of the hyperfine atomic structure.

The dynamic case will be treated similarly elsewhere.

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